Suggested solutions to Midterm test for MATH3270a

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1. (**20points=2 points $\times$ 10**) **Mark** each of the following statement **True** (meaning that it is a true statement) or **False** (meaning that it is a false statement), no expanation is needed.

   (a) The equation $y'' + t^2y' + y = 3$ is a linear differential equation;

   (b) $\sin(t^3)$ can be a solution to a differential equation of the form $y'' + p(t)y' + q(t)y = 0$ with continuous $p(t), q(t)$ on $\mathbb{R}$.

   (c) For any continuous function $f(t)$ on $I = \mathbb{R}$, then there exists a unique solution to the initial value problem $f(t)y' = y, y(0) = 0$.

   (d) The second order equation $y'' + 3y' + y = \cos 6t$ has a particular solution of the form $Y(t) = A\cos 6t$.

   (e) Consider $M(t, y) = \frac{-t}{t^2+y^2}$ and $N = \frac{t}{t^2+y^2}$ defining on $\mathbb{R}^2\backslash\{0\}$, there exists a function $\psi(t, y)$ such that $\frac{\partial\psi}{\partial t} = M$ and $\frac{\partial\psi}{\partial y} = N$ on $\mathbb{R}^2\backslash\{0\}$.

   (f) The function $2ty^3 + 3y\cos(ty) + (3t^2y^2 + 3t\cos(ty))y' = 0$ is an exact equation.

   (g) Consider the second order equation $y'' + p(t)y' + q(t)y = 0$ (with continuous $p(t)$ and $q(t)$) with two solutions y_1 and y_2 defining on interval I, then y_1 and y_2 linearly independent if and only if $W(y_1, y_2)(t) \neq 0$ on the whole I.

   (h) If y_1 and y_2 are solutions to the equation $y'' + p(t)y' + q(t)y = r(t)$ with $r(t) \neq 0$, then $y = c_1y_1 + c_2y_2$ is still a solution to $y'' + p(t)y' + q(t)y = r(t)$ for any $c_1, c_2 \in \mathbb{R}$.

   (i) The second order equation $y'' + 2y' + y = e^{-t}$ has a solution of the form $Y(t) = At^2e^{-t}$.

   (j) Suppose y_1 and y_2 are solutions to the equation $y''+p(t)y'+q(t)y = 0$, then there Wronskian $W(t) = W(y_2, y_2)(t)$ satisfies the differential equation $W'(t) + p(t)W(t) = 0$.

   Solutions:

   (a) True;

   (b) False;

   (c) False;

   (d) False;

   (e) False;

   (f) True;

   (g) True;

   (h) False;

   (i) True;

   (j) True.

2. (**30points=10 points $\times$ 3**) **Solve** the general solution for the following order equations:

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(a) $ty' + (1+t)y = e^{-t}\sin 2t, \quad$ for $t > 0$;

(b) $(2y\sin t - t^3 + \log y)y' = -(y^2\cos t - 3t^2 y - 2t)$;

(c) $y'' + \frac{4}{t}y' + \frac{2}{t^2}y = 0, \quad$ for $t > 0$.

Solution:

(a) Multiplying the ODE by e^t gives

$$\frac{d}{dt}(te^t y) = \sin 2t,$$

then integrating both sides we have

$$te^t y = -\frac{1}{2}\cos 2t + C$$

or equivalently,

$$y = -\frac{1}{2t}e^{-t}\cos 2t + C\frac{1}{t}e^{-t}$$

where C is an arbitrary constant.

(b) Rewrite the ODE as

$$(y^2\cos t - 3t^2 y - 2t) + (2y\sin t - t^3 + \log y)y' = 0.$$

Let $M = y^2\cos t - 3t^2 y - 2t$ and $N = 2y\sin t - t^3 + \log y$, then

$$M_y = 2y\cos t - 3t^2 = N_t$$

which implies that it is exact. Thus there exists a function $\varphi(t,y)$ such that

$$\partial_t\varphi = M = y^2\cos t - 3t^2 y - 2t, \tag{1}$$
$$\partial_y\varphi = N = 2y\sin t - t^3 + \log y. \tag{2}$$

By solving (1) firstly, we have

$$\varphi(t,y) = y^2\sin t - t^3 y - t^2 + h(y) \tag{3}$$

with some function $h(y)$. Then insert (3) into (2), we have

$$h'(y) = \log y$$

which implies that

$$h(y) = \int \log y dy = y\log y - y + C.$$

Finally, the general solution to the ODE is given by

$$\varphi = y^2\sin t - t^3 y - t^2 + y\log y - y = C$$

with arbitrary constant C.

(c) It's noted that it's a Euler-type ODE, so it's promising to find the solution with the following form

$$Y = t^\alpha$$

with constant α to be determined. So $Y' = \alpha t^{\alpha-1}, Y'' = \alpha(\alpha-1)t^{\alpha-2}$. If Y is indeed a solution, then

$$\alpha(\alpha-1)t^{\alpha-2} + \frac{4}{t}\alpha t^{\alpha-1} + \frac{2}{t^2}t^{\alpha} = 0$$

which implies that

$$\alpha^2 + 3\alpha + 2 = 0.$$

and that

$$\alpha_1 = -1, \alpha_2 = -2.$$

Thus $y_1 = t^{-1}$ and $y_2 = t^{-2}$ are two solutions, and it's easy to check that they form a fundamental set of solution, so the general solution is given by

$$y = C_1 t^{-1} + C_2 t^{-2}$$

where C_1, C_2 are arbitrary constants.

3. (**15points**)Consider the inhomogeneous differential equation

$$y'' + 4y = 8t^2 + 10e^{-t}.$$

   (a) (**5points**) **Find** a fundamental set of solution to the corresponding homogeneous equation.

   (b) (**10points**) **Find** the general solution to the above inhomogeneous equation.

Solution:

(a) The corresponding characteristic equation is

$$r^2 + 4 = 0,$$

then $r = \pm 2i$, so a fundamental set of solution is given by

$$\{\cos 2t, \sin 2t\}.$$

(b) First, we find a particular solution.
Method1: Undetermined coefficients
By observation, it's promising to find a particular solution of the form

$$Y(t) = At^2 + Bt + C + De^{-t}$$

where constants A, B, C, D are to be determined. Since

$$\begin{aligned} Y'(t) &= 2At + B - De^{-t}, \\ Y''(t) &= 2A + De^{-t}, \end{aligned}$$

then by substituting $Y(t)$ into the problem, we have

$$\begin{aligned} 8t^2 + 10e^{-t} &= Y'' + 4Y \\ &= 4At^2 + 4Bt + 4C + 2A + 5De^{-t} \end{aligned}$$

which implies that $A = 2, B = 0, C = -1, D = 2$. So a particular solution is given by

$$Y = 2t^2 - 1 + 2e^{-t}.$$

Method2: Variation of parameters

Let $y_1 = \cos 2t, y_2 = \sin 2t$ and then $W(y_1, y_2) = 2$, then a particular solution of homogeneous problem with $r(t) = 8t^2 + 10e^{-t}$ is given by

$$\begin{aligned} Y(t) &= -y_1(t) \int \frac{y_2(t)r(t)}{W(y_1, y_2)} dt + y_2(t) \int \frac{y_1(t)r(t)}{W(y_1, y_2)} dt \\ &= -\cos 2t \int 4t^2 \sin 2t + 5e^{-t} \sin 2t \ dt + \sin 2t \int 4t^2 \cos 2t + 5e^{-t} \cos 2t \ dt \\ &= 2t^2 - 1 + 2e^{-t}. \end{aligned}$$

Finally, the general solution is given by

$$y = C_1 \cos 2t + C_2 \sin 2t + 2t^2 - 1 + 2e^{-t}$$

with arbitrary constants C_1 and C_2.

4. (**20points**) Consider the 2^{nd} order inhomogeneous differential equation

$$y'' + (1 - \frac{2}{t})y' - (\frac{t-2}{t^2})y = 2t,$$

for $t > 0$.

(a) (**12points**) Given a solution $y_1(t) = t$ to the corresponding homogeneous equation

$$y'' + (1 - \frac{2}{t})y' - (\frac{t-2}{t^2})y = 0,$$

find another solution $y_2(t)$ to the homogeneous equation such that they form a fundamental set of solutions, by **computing** their Wronskian $W(y_1, y_2)(t)$ and show that it is non-zero.

(b) (**8points**) **Find** the general solutions to the above inhomogeneous differential equation.

Solution:

(a) It follows from Abel's theorem that the Wronskian of any two solutions to homogeneous equation $y'' + p(t)y' + q(t)y = 0$ satisfies

$$W'(y_1, y_2)(t) = -p(t)W(y_1, y_2)(t)$$

which implies that

$$W(y_1, y_2)(t) = C_1 e^{-\int p(t)dt}.$$

where $C_1 \in \mathbb{R}$ is a constant. Here, $p(t) = 1 - \frac{2}{t}$, let $y_1 = t$ and y_2 is another solution to homogeneous equation such that y_1, y_2 form a fundamental set of solution, that is $W(y_1, y_2) \neq 0$, so without loss of generality assume that $C_1 = -1$, then

$$ty_2' - y_2 = W(t, y_2) = -e^{-\int 1 - \frac{2}{t} dt} = -t^2 e^{-t}.$$

This is a first order ODE of y_2, multiplying by $\frac{1}{t^2}$ on both sides gives

$$\left(\frac{y_2}{t}\right)' = -e^{-t}$$

and then integrating yields

$$y_2 = te^{-t} + C_2 t$$

where C_2 is an arbitrary constant. In particular, we can take $y_2(t) = te^{-t}$.

(b) Let $y_1 = t, y_2 = te^{-t}$ and then $W(y_1, y_2) = -t^2e^{-t}$, then a particular solution of homogeneous problem with $r(t) = 2t$ is given by

$$\begin{aligned} Y(t) &= -y_1(t)\int \frac{y_2(t)r(t)}{W(y_1,y_2)}dt + y_2(t)\int \frac{y_1(t)r(t)}{W(y_1,y_2)}dt \\ &= 2t\int dt - 2te^{-t}\int e^t dt \\ &= 2t^2 - 2t \end{aligned}$$

Hence, the general solution to inhomogeneous equation is

$$y = 2t^2 + C_1t + C_2te^{-t}$$

where C_1, C_2 are arbitrary constant.

5. (**15points**) Consider the 2^{nd} order homogeneous differential equation

$$y'' + p(t)y' + q(t)y = 0$$

with two linearly independent solutions y_1, y_2 on an interval I.

(a) (**4points**) **Show** that y_1 and y_2 cannot be simultaneous zero at the same point in I.

(b) (**11points**) Assuming $y_1(a) = y_1(b) = 0$ for some $a < b$ and $y_1(t) \neq 0$ for other $a < t < b$, **show** that there exist is a unique t_0 satisfying $a < t_0 < b$ such that $y_2(t_0) = 0$.

Solution:

(a) Since y_1, y_2 are two linearly independent solutions, so they form a fundamental set of solution to the problem, then $W(y_1, y_2)(t) \neq 0$ for any $t \in I$ by Abel's theorem.
Suppose there is a point $c \in I$ such that $y_1(c) = y_2(c) = 0$, then

$$W(y_1, y_2)(c) = 0,$$

which is impossible.

(b) **Existence**. Assume that $y_2 \neq 0$ on the interval $(a, b) \subset I$, and thus

$$\begin{aligned} &y_1(a) = y_1(b) = 0, \\ &y_2(t) \neq 0, \quad a \le t \le b. \end{aligned}$$

Let $v(t) = \frac{y_1}{y_2}, t \in [a, b]$, so $v(a) = v(b) = 0$, then

$$v' = \frac{y_1'y_2 - y_1y_2'}{y_2^2} = \frac{W(y_1, y_2)}{y_2^2}$$

which implies that

$$\frac{y_1}{y_2} = v(t) = \int_a^t \frac{W(y_1,y_2)(s)}{y_2^2(s)}ds$$

or

$$y_1 = y_2(t)v(t) = y_2(t)\int_a^t \frac{W(y_1,y_2)(s)}{y_2^2(s)}ds.$$

In particular, $t = b$, we have

$$y_1(b) = y_2(b)\int_a^b \frac{W(y_1,y_2)(s)}{y_2^2(s)}ds = 0$$

which implies that

$$W(y_1, y_2)(t) \equiv 0, \quad a \le t \le b.$$

Thus there exists a point $t_0 \in (a, b)$ such that $y_2(t_0) = 0$.
Uniqueness. Suppose that there exists another $t_1 \in (a, b)$ such that $t_1 \neq t_0, y_2(t_1) = 0$. Since

$$\begin{aligned}
&y_2(t_0) = y_2(t_1) = 0 \\
&y_1(t) \neq 0, \quad \min\{t_0, t_1\} \le t \le \max\{t_0, t_1\},
\end{aligned}$$

we can define

$$V(t) = \frac{y_2}{y_1}, \quad t \in [\min\{t_0, t_1\}, \max\{t_0, t_1\}],$$

and apply the existence argument for $V(t)$, then there at least exists one point $t_* \in [\min\{t_0, t_1\}, \max\{t_0, t_1\}] \subset (a, b)$ such that $y_1(t_*) = 0$, which contradicts with the assumption that $y_1(t) \neq 0$ for $a < t < b$.